

ANALYTICAL SOLUTION OF REGULARIZED LONG WAVE (RLW) EQUATION WITH HOMOTOPY ANALYSIS METHOD

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Abstract

In this paper, the analytic solution of regularized long wave equation is developed via homotopy analysis method (HAM). This method is strong and easy to use analytic tool for investigating of nonlinear problems. Homotopy analysis method contains the auxiliary parameter \hbar , which provides us with a simple way to adjust and control the convergence region of solution series. Our work reveals that homotopy analysis method is very convenient and effective. In comparison with HAM, numerical methods leads to inaccurate results when the equation is intensively dependent on time, while (HAM) overcome the above shortcomings completely and therefore can be widely applied in engineering problems. By using an initial value system, the numerical results of regularized long wave equation have been represented graphically.

1. Introduction

Nonlinear phenomena play important roles in applied mathematics, physics, and also in engineering problems in which each parameter

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varies depending on different factors. Some of the relevant areas to wave phenomena contain solid state physics [6], fluid dynamics [27], chemical kinetics [10], plasma physics [11, 22], elastic media, optical fibers etc. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose, which has been quite untouched.

In recent years, several such techniques have drawn special attention, such as Hirota's bilinear method [16], the homogeneous balance method [9, 26], inverse scattering method [25], the Adomian decomposition method (ADM) [2, 3], the variation iteration method [13], the homotopy perturbation method (HPM) [14, 15], and the homotopy analysis method. The homotopy analysis method [17-19], which was introduced by Liao, is one of the most effective techniques among these methods, which has been successfully applied to solve many types of nonlinear problems by others [1, 12, 23].

The motivation of this paper is to extend the (HAM) to solve regularized long wave (RLW) equation. This equation is one of the important nonlinear wave equations. A large number of important physical phenomena such as shallow water waves and plasma waves [20] can be described by this equation. Numerical solutions based on finite difference technique [24], Runge-Kutta and predictor/corrector methods [4], collocation method [5], Galerkin's method [8], and Adomian decomposition method [7] were reported in the open literature. RLW equation derived for long waves propagating in the positive x -direction has the form, [21];

$$\frac{\partial}{\partial t} u(x, t) + (1 + u(x, t)) \frac{\partial}{\partial x} u(x, t) - \gamma \frac{\partial^3}{\partial x^2 \partial t} u(x, t) = 0, \quad (1)$$

where γ is a positive parameter and the subscripts x and t denote differentiation, with the boundary condition $u \rightarrow 0$ as $x \rightarrow \infty$.

The soliton solution of RLW equation has the form of [7];

$$u(x, t) = 3c \sinh^2[p(x - 9t - x_0)], \quad (2)$$

where

$$p = \sqrt{\frac{c}{4\gamma(c+1)}}, \quad 9 = c + 1, \quad (3)$$

and c is a constant.

2. The Basic Idea of Homotopy Analysis Method

Let us assume the following nonlinear differential equation in form of

$$N[u(\tau)] = 0, \quad (4)$$

where N is a nonlinear operator, τ is an independent variable, and $u(\tau)$ is the solution of equation. We define the function, $\phi(\tau, p)$ as follows:

$$\lim_{p \rightarrow 0} \phi(\tau, p) = u_0(\tau), \quad (5)$$

where $p \in [0, 1]$ and $u_0(\tau)$ is the initial guess, which satisfies the initial or boundary conditions and

$$\lim_{p \rightarrow 1} \phi(\tau, p) = u(\tau), \quad (6)$$

and by using the generalized homotopy method, Liao's so-called zero-order deformation equation (4) will be

$$(1 - p)L[\phi(\tau, p) - u_0(\tau)] = p\hbar H(\tau)N[\phi(\tau, p)], \quad (7)$$

where \hbar is the auxiliary parameter, which helps us to increase the results convergence, $H(\tau)$ is the auxiliary function, and L is the linear operator. It should be noted that, there is a great freedom to choose the auxiliary parameter \hbar , the auxiliary function $H(\tau)$, the initial guess $u_0(\tau)$, and the auxiliary linear operator L . This freedom plays an important role in establishing the keystone of validity and flexibility of

HAM as shown in this paper. Thus, when p increases from 0 to 1, the solution $\phi(\tau, p)$ changes between the initial guess $u_0(\tau)$ and the solution $u(\tau)$. The Taylor series expansion of $\phi(\tau, p)$ with respect to p is

$$\phi(\tau, p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) p^m, \quad (8)$$

and

$$u_0^{[m]}(\tau) = \left. \frac{\partial^m \phi(\tau, p)}{\partial p^m} \right|_{p=0}, \quad (9)$$

where $u_0^{[m]}(\tau)$ for brevity is called the *m-th order of deformation derivation*, which reads

$$u_m(\tau) = \frac{u_0^{[m]}(\tau)}{m!} = \frac{1}{m!} \left. \frac{\partial^m \phi(\tau, p)}{\partial p^m} \right|_{p=0}. \quad (10)$$

It is clear that, if the auxiliary parameter $\hbar = -1$ and auxiliary function $H(\tau) = 1$, then Equation (4) will become

$$(1-p)L[\phi(\tau, p) - u_0(\tau)] + p(\tau)N[\phi(\tau, p)] = 0. \quad (11)$$

This statement is commonly used in HPM procedure. Indeed, in HPM, we solve the nonlinear differential equation by separating any Taylor expansion term. Now, we define the vector of

$$\vec{u}_m = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n\}. \quad (12)$$

According to the definition in Equation (10), the governing equation and the corresponding initial conditions of $u_m(\tau)$ can be deduced from zero-order deformation equation (4). Differentiating the Equation (4) m times with respect to the embedding parameter p and setting $p = 0$ and finally, dividing by $m!$, we will have the so-called *m-th-order deformation equation* in the form

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R(\vec{u}_{m-1}), \quad (13)$$

where

$$R_m(\bar{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(\tau, p)]}{\partial p^{m-1}} \right|_{p=0}, \quad (14)$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \quad (15)$$

So by applying inverse linear operator to both sides of the linear equation, Equation (4), we can easily solve the equation and compute the generation constant by applying the initial or boundary condition.

3. Application

Following the homotopy analysis method, the nonlinear operator is defined as

$$\begin{aligned} N[\phi(x, t, p)] \\ = \frac{\partial}{\partial t} u(x, t, p) + (1 + u(x, t, p)) \frac{\partial}{\partial x} u(x, t, p) - \gamma \frac{\partial^3}{\partial x^2 \partial t} u(x, t, p). \end{aligned} \quad (16)$$

And linear operator is defined as

$$L[\phi(x, t, p)] = \frac{\partial}{\partial t} u(x, t, p). \quad (17)$$

For application of the method, we choose the initial approximation

$$u_0(x, t) = 3 \operatorname{sech}^2(x). \quad (18)$$

Using the above definition, with assumption $H(x, t) = 1$, we construct the zero-order deformation equation

$$(1-p)L[\phi(x, t, p) - u_0(x, t)] = p\hbar N[\phi(x, t, p)]. \quad (19)$$

Obviously, when $p = 0$ and $p = 1$,

$$\phi(x, t, 0) = u_0(x, t), \quad \phi(x, t, 1) = u(x, t). \quad (20)$$

Expanding $\phi(x, t, p)$ in Taylor series with respect to p , we have

$$\phi(x, t, p) = u_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t)p^m, \quad (21)$$

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \phi(x, t, p)|_{p=0}. \quad (22)$$

Note that the convergence regions of the series (21) is dependent upon the auxiliary parameters \hbar . If this auxiliary parameter is properly chosen so that series (21) is convergent at $p = 1$, therefore using Equation (20), we have

$$\phi(x, t, p) = u_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t)p^m. \quad (23)$$

Differentiating the Equation (19) m times with respect to p , and then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called m -th order of deformation equation for $f_m(\eta)$ and $\theta_m(\eta)$;

$$L[u_m(x, t) - X_m u_{m-1}(x, t)] = \hbar R_m(u_{m-1}(x, t)). \quad (24)$$

Subject to the boundary conditions

$$u_m(x, t) = 0. \quad (25)$$

Under the definitions,

$$R_m(u_{m-1}) = \frac{\partial}{\partial t} u_{m-1} + \frac{\partial}{\partial x} u_{m-1} + \sum_{j=1}^{m-1} u_j \cdot \frac{\partial}{\partial x} u_{m-1-j} - \gamma \frac{\partial^3}{\partial x^2 \partial t} u_{m-1}, \quad (26)$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \quad (27)$$

Now, the solution of the m -th-order deformation equation (24) from $m \geq 1$, becomes

$$u_m(x, t) = X_m u_{m-1}(x, t) - \hbar L^{-1} R_m(u_{m-1}(x, t)). \quad (28)$$

From (18), (28), we now successively obtain

$$u_1(x, t) = -\frac{1.2t[\cosh^2(x) + 3]\sinh(x)}{\cosh^5(x)}, \quad (29)$$

$$u_2(x, t) = \dots,$$

$$u_3(x, t) = \dots$$

The solution of Equation (13) in series form is given by

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots + u_n(x, t). \quad (30)$$

We can approximate a nonlinear problem more efficiently by choosing a proper set of base functions and ensure its convergence. Note that this series contains the auxiliary parameter \hbar , which influence its convergence region and rate. We should ensure that the solution converges. Note that we still have freedom to choose the auxiliary parameter \hbar , as pointed by Liao [18], the convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter \hbar .

We first plot the so-called h -curve of $u_t(4, 0.9)$ by 7-th-order approximation of solution, as shown in Figure 1, it is easy to discover the valid region of \hbar . According to Figure 1, it is seen that convergent results can be obtained $0 < \hbar_1 < 0.5$.

We have developed solution up to 7-th order of approximation of $U(x, t)$ as you can see in Figures 1, 2 and Table 1.

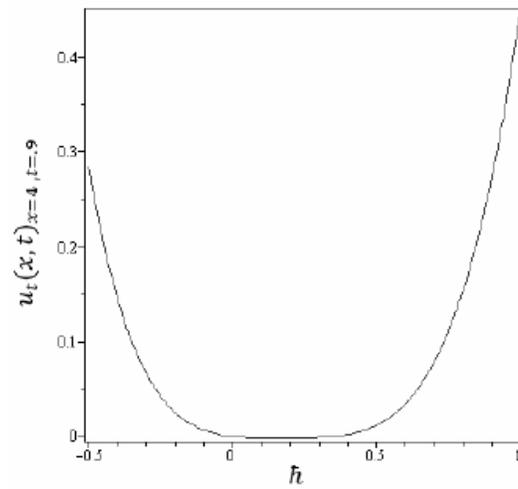


Figure 1. h for $u_t(x, t)$ by 7-th order approximation.

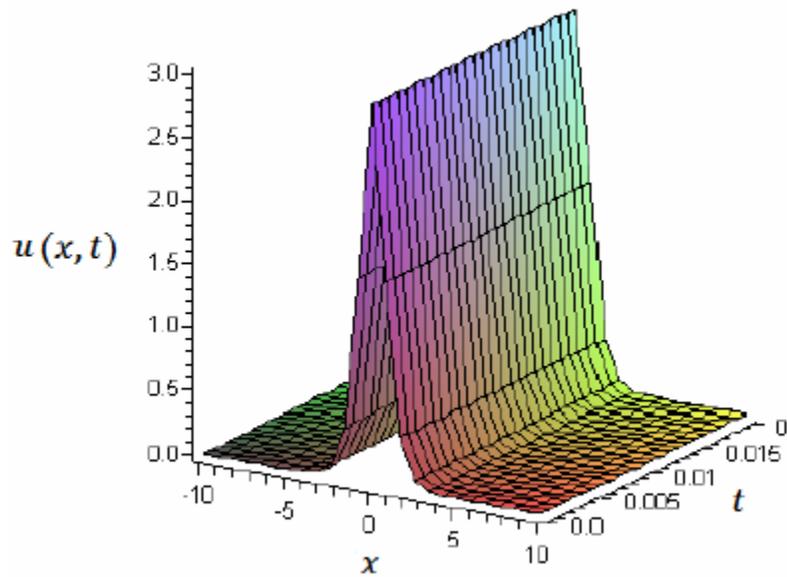


Figure 2. The behavior of the analytic solution $u(x, t)$ versus x for different values of time with HAM method.

Table 1. Numerical comparison between homotopy analysis method (HAM) and the Adomian decomposition method (ADM) [7] at various values of (x, t)

t	x	HAM	ADM
0.001	-5.0	0.0005425352533	0.0005425752432
	-2.5	0.07946279935	0.07946245674
	0	2.999948864	2.99998800
	2.5	0.08000764839	0.0800912881
	5	0.0005469327148	0.0005469328566
0.002	-5.0	0.000540409397	0.0005404094721
	-2.5	0.07914945652	0.07914945435
	0	2.999952045	2.999952001
	2.5	0.08040880438	0.08040880776
	5	0.00054912431	0.0005491247682
0.003	-5.0	0.00053825232	0.0005382523449
	-2.5	0.0788376612	0.07883766831
	0	2.999895792	2.999892003
	2.5	0.08072674785	0.08072672151
	5	0.0005513965049	0.0005513254644
0.004	-5.0	0.0005361954360	0.0005361038280
	-2.5	0.07852768485	0.07852709388
	0	2.999808871	2.999808009
	2.5	0.08104507868	0.08104587477
	5	0.0005535332720	0.0005535349782
0.005	-5.0	0.0005396833323	0.0005339638866
	-2.5	0.07821775624	0.07821772659
	0	2.99970013	2.999700022
	2.5	0.07933239680	0.08136627222
	5	0.0005557533196	0.0005557533474
0.01	-5.0	0.0005233916020	0.0005233916154
	-2.5	0.0766884008	0.07668883740
	0	2.998800183	2.998800318
	2.5	0.08298709194	0.08298708768
	5	0.0005669792497	0.0005669792487

4. Conclusion

In this paper, the homotopy analysis method has been successfully applied to finding the solution of the regularized long wave (RLW) equation. This method solves the problem without any need to discretize the variables. Therefore, it is not affected by computation round-off errors and one does not face the necessity of large computer memory and time. This method is useful for finding an accurate approximation of the exact solution. The results show that the homotopy analysis method is a promising method to solve nonlinear equations.

References

- [1] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, *Phys. Lett. A* 360 (2006), 109-113.
- [2] G. Adomian, A review of the decomposition method in applied mathematics, *J. Math. Anal. Appl.* 135 (1988), 501-544.
- [3] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, MA, 1994.
- [4] J. L. Bona, W. G. Pritchard and L. R. Scott, Numerical scheme for a model of nonlinear dispersive waves, *J. Comput. Phys.* 60 (1985), 167-176.
- [5] I. Dag, Application of cubic *B*-splines for numerical solution of the RLW equation, *Appl. Math. Comput.* 159 (2004), 373-389.
- [6] G. Eilenberger, *Solitons, Mathematical Methods for Physicists*, Springer-Verlag, 1983.
- [7] T. S. El-Danaf, M. A. Ramadan and F. E. I. Abd Alaal, The use of Adomian decomposition method for solving the regularized long-wave equation, *Chaos, Solitons and Fractals* 26 (2005), 747-757.
- [8] A. Esen and S. Kutluay, Application of a lumped Galerkin method to the regularized long wave equation, *Appl. Math. Comput.* 174 (2006), 833-845.
- [9] E. Fan, Two new applications of the homogeneous balance method, *Physics Letters A* 256 (2000), 353-357.
- [10] P. Gray and S. Scott, *Chemical Oscillations and Instabilities*, Clarendon, Oxford, 1990.
- [11] A. Hasegawa, *Plasma Instabilities and Nonlinear Effects*, Springer-Verlag, Berlin, 1975.
- [12] T. Hayat, M. Khan and S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, *Acta Mech.* 168 (2004), 213-232.

- [13] J. H. He, Variational iteration method for autonomous ordinary differential systems, *Appl. Math. Comput.* 114 (2000), 115-123.
- [14] J. H. He, Comparison of homotopy perturbation and homotopy analysis method, *Appl. Math. Comput.* (2004), 156-527.
- [15] J. H. He, Some asymptotic methods for strongly nonlinear equations, *Int. J. Mod. Phys.* 20 (2006), 1141-1199.
- [16] X. B. Hu and Y. T. Wu, Application of the Hirota bilinear formalism to a new integrable-difference equation, *Physics Letters A* 246 (1998).
- [17] S. J. Liao, On the Proposed Homotopy Analysis Technique for Nonlinear Problems and its Applications, Ph.D. Dissertation, Shanghai Jiao Tong University, 1992.
- [18] S. J. Liao, Numerically solving nonlinear problems by the homotopy analysis method, *Comput. Mech.* 20 (1997), 530-540.
- [19] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, CRC Press, Chapman and Hall, Boca Raton 17 (2003), 1-16.
- [20] D. H. Pregrine, Calculations of the development of an undular bore, *J. Fluid Mech.* 25 (1996), 321-330.
- [21] K. R. Raslan, A computational method for the regularized long wave equation, *Appl. Math. Comput.* 167 (2005), 1101-1118.
- [22] S. Sasha Ray, A numerical solution of coupled sine-Gordon equation by using the modified decomposition method, *Appl. Math. Comput.* 175 (2006).
- [23] M. Sajid, T. Hayat and S. Asghar, Comparison between HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt, *Nonlinear Dyn.* 50 (2007), 27-35.
- [24] S. I. Saki, Solitary waves of RLW equation, *Comput. Phys. Commun.* 138 (2001), 80-91.
- [25] V. O. Vakhnenko, E. J. Parkes and A. J. Mpprison, A Bäcklund transformation and the inverse scattering transformation method for the generalized Vakhnenko equation, *Chaos, Solitons and Fractals* 17 (2003), 683-692.
- [26] M. Wang, Y. Zhou and Z. Li, Applications of homogeneous balance method to exact solution of nonlinear equations in mathematical physics, *Physics Letters A* 216 (1996), 67-75.
- [27] G. Whitham *Waves*, Wiley, Linear and Nonlinear, New York, 1974.

